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Dirac Oscillator in Noncommutative Phase Space

Shaohong Cai · Tao Jing · Guangjie Guo · Rukun Zhang

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Abstract We study the Dirac oscillators in a noncommutative phase space. The results show that the energy gap of Dirac oscillator was changed by noncommutative effect. In addition, we obtain the non-relativistic limit of the energy spectrum.

Keywords Noncommutative phase space · Dirac oscillator · Exact solutions

1 Introduction

In the last few years theories in noncommutative space have been studied extensively. Noncommutative field theories are related to M-theory compactification [1], string theory in nontrivial backgrounds [2] and quantum Hall effect [3]. A simple insight on the role of noncommutativity in field theory can be obtained by studying the one particle sector, which prompted an interest in the study of noncommutative quantum mechanics (NCQM) [4–10]. In these studies some attention was paid to two-dimensional NCQM and its relation to the landau problem. It has been shown that the equation of motion of a harmonic oscillator in a noncommutative space is similar to the equation of motion of a particle in a constant magnetic field and in the lowest Landau level. Dirac relativistic oscillator is an important potential both for theory and application, which was the first time studied by Ito et al. [11]. A lot of papers have recently published concerning the solution and properties of the Dirac equation with the Dirac relativistic oscillator in ordinary commutative space [12–18].

As we know, in the domain of relativistic extension NCQM, Dirac oscillator and Klein Gordon oscillator has been discussed by Mirza et al. in noncommutative space [19], and so was DKP equation with a relativistic oscillator for spin 0 and spin 1 by Falek et al. [20]. In

S. Cai
Guizhou College of Finance and Economics, Guiyang Guizhou 550004, China
e-mail: caish@gzife.edu.cn

T. Jing (✉) · G. Guo · R. Zhang
Department of Physics, Guizhou University, Guiyang Guizhou 550025, China
e-mail: gs.jingt@mail.gzu.edu.cn

this note, we shall consider a where both position and momentum coordinates are noncommutative. The reason for taking the momentum noncommutativity comes from the fact that in quantum mechanics, the generalized momentum components are noncommutative. And it has been shown recently [21] that in order to keep the Bose-Einstein statistics for identical particles intact at the noncommutative level, one should consider both the space-space and momentum-momentum noncommutativity. In addition, we discussed the non-relativistic limit of the energy spectrum.

First of all, let us review some usefu1 formulas. The first-order relativistic Dirac equation

$$\left[c\vec{\alpha}.\bar{p}+\beta mc^2\right]\psi(\vec{r})=E\psi(\vec{r}), \tag{1}$$

where α_i $(i=1,2,3)$ matrices satisfy the commutation relation

$$\begin{aligned}&\alpha_i^2=\beta^2=1,\\&\alpha_i\alpha_j+\alpha_j\alpha_i=0,\\&\alpha_i\beta+\beta\alpha_i=0,\end{aligned}$$

where

$$\vec{\alpha}=\begin{pmatrix}0&\vec{\sigma}\\\vec{\sigma}&0\end{pmatrix},\qquad \beta=\begin{pmatrix}I&0\\0&-I\end{pmatrix}. \tag{2}$$

Considering the symmetry of the linear transformation between the noncommutative variables and the commutative variables, we choose the commutation relations introduced in reference [22]

$$[\hat{x},\hat{y}]=i\theta,\qquad [\hat{p}_x,\hat{p}_y]=i\eta,\qquad [\hat{x},\hat{p}_x]=i\hbar_{eff},\qquad [\hat{y},\hat{p}_y]=i\hbar_{eff}, \tag{3}$$

where $\hbar_{eff}=\hbar(1+\frac{\theta\eta}{4\hbar^2})$, and θ,η are positive real parameters respectively reflecting the noncommutativity of coordinates and momenta. By comparison between theoretical predictions for some specific noncommutative systems and experimental data, one can find bounds to these noncommutative parameters [22–24] as $\theta\leq 4\times10^{-40}$ m^2 and $\eta\leq 1.76\times10^{-61}$ kg^2 m^2 s^{-2}. It follows that $\frac{\theta\eta}{4\hbar^2}\cong O(10^{-37})$ and hence the correction to Planck constant is irrelevant.

We can obtain (3) by using the following linear transformation between noncommutative variables and commutative variables

$$\hat{x}=x-\frac{\theta}{2\hbar}p_y,\qquad \hat{y}=y+\frac{\theta}{2\hbar}p_x,\qquad \hat{p}_x=p_x+\frac{\eta}{2\hbar}y,\qquad \hat{p}_y=p_y-\frac{\eta}{2\hbar}x, \tag{4}$$

where the operators x,y,p_x,p_y in commutative space satisfy ordinary Heisenberg commutation relations

$$[x,p_x]=i\hbar,\qquad [y,p_y]=i\hbar, \tag{5}$$

and other commentators of these operators are vanishing.

2 Dirac Oscillators in a Noncommutative Phase Space

The Dirac oscillator in a commutative space is defined by the following substitution suggested by Ito et al. [11]

$$p_i\to p_i-i\beta m\omega x_i. \tag{6}$$

In two dimensions, this problem is exactly solvable. In this case the Dirac oscillator can be written as

$$\left[c\vec{\alpha}\cdot(\vec{p}-i\beta m\omega\vec{r})+\beta mc^2\right]\psi(\vec{r})=E\psi(\vec{r}). \tag{7}$$

Since we are dealing with two component spinors it is convenient to introduce the following representation in terms of the Pauli matrices [25]:

$$\psi(\vec{r})=\begin{pmatrix}\psi_A(\vec{r})\\ \psi_B(\vec{r})\end{pmatrix}, \tag{8}$$

$$\alpha_x=\sigma_1,\qquad \alpha_y=s\sigma_2,\qquad \beta=\sigma_3, \tag{9}$$

where the parameter s takes the values ±1 ($+1$ for spin up and -1 for spin down).

The two dimensional Dirac equation can be separated to two equations

$$c\left[\left(p_x-isp_y\right)+im\omega\left(x-isy\right)\right]\psi_B=\left(E-mc^2\right)\psi_A, \tag{10}$$

$$c\left[\left(p_x+isp_y\right)-im\omega\left(x+isy\right)\right]\psi_A=\left(E+mc^2\right)\psi_B. \tag{11}$$

The Dirac oscillator in a noncommutative phase space is given by the following equation

$$\left[c\vec{\alpha}\cdot(\vec{p}-i\beta m\omega\vec{r})+\beta mc^2\right]*\psi(\vec{r})=E\psi(\vec{r}).$$

The two dimensional Dirac oscillator in a noncommutative phase space can be separated to two equations

$$c[(\hat{p}_x-is\hat{p}_y)+im\omega(\hat{x}-is\hat{y})]\psi_B=(E-mc^2)\psi_A, \tag{12}$$

$$c\left[\left(\hat{p}_x+is\hat{p}_y\right)-im\omega\left(\hat{x}+is\hat{y}\right)\right]\psi_A=\left(E+mc^2\right)\psi_B. \tag{13}$$

Using the new coordinates (4) in a commutative space, we can map the Dirac oscillator in a noncommutative phase space to a commutative one,

$$\begin{aligned}&c\left[\left(p_x+\frac{\eta}{2\hbar}y\right)-is\left(p_y-\frac{\eta}{2\hbar}x\right)+im\omega\left[\left(x-\frac{\theta}{2\hbar}p_y\right)-is\left(y+\frac{\theta}{2\hbar}p_x\right)\right]\right]\psi_B\\&\quad=(E-mc^2)\psi_A,\end{aligned} \tag{14}$$

$$\begin{aligned}&c\left[\left(p_x+\frac{\eta}{2\hbar}y\right)+is\left(p_y-\frac{\eta}{2\hbar}x\right)-im\omega\left[\left(x-\frac{\theta}{2\hbar}p_y\right)+is\left(y+\frac{\theta}{2\hbar}p_x\right)\right]\right]\psi_A\\&\quad=(E+mc^2)\psi_B.\end{aligned} \tag{15}$$

By comparing (10) and (14) with the Landau problem in non-relativistic quantum mechanics,one finds that the Dirac oscillators in noncommutative phase-space have similar properties as the dynamics of a particle in a uniform magnetic field in a commutative space.

After straightforward calculation we arrive at the following equations

$$\begin{aligned}&c^2\left[\left(1+\frac{sm\omega\theta}{2\hbar}\right)^2(p_x^2+p_y^2)+m^2\omega^2\left(1+\frac{s\eta}{2\hbar m\omega}\right)^2(x^2+y^2)\right.\\&\quad\left.-2m\omega\left(1+\frac{sm\omega\theta}{2\hbar}\right)\left(1+\frac{s\eta}{2\hbar m\omega}\right)\left(sL_z+2\left(\frac{\hbar}{2}\right)\right)\right]\psi_A=(E^2-m^2c^4)\psi_A,\end{aligned} \tag{16}$$

and

$$c^2\left[\left(1+\frac{sm\omega\theta}{2\hbar}\right)^2(p_x^2+p_y^2)+m^2\omega^2\left(1+\frac{s\eta}{2\hbar m\omega}\right)^2(x^2+y^2)\right.$$
$$\left.-2m\omega\left(1+\frac{sm\omega\theta}{2\hbar}\right)\left(1+\frac{s\eta}{2\hbar m\omega}\right)\left(sL_z+2\left(-\frac{\hbar}{2}\right)\right)\right]\psi_B=(E^2-m^2c^4)\psi_B, \quad (17)$$

(16) and (17) can be written

$$\left[\frac{p_x^2+p_y^2}{2M}+\frac{1}{2}M\Omega^2(x^2+y^2)-D(p_yx-p_xy)\right]\psi=\tilde{E}\psi, \quad (18)$$

where

$$M(\theta,s)=m\frac{1}{(1+\frac{sm\omega\theta}{2\hbar})^2}, \qquad \frac{1}{2}M\Omega^2(\theta,\eta,s)=\frac{m\omega^2}{2}\left(1+\frac{s\eta}{2\hbar m\omega}\right)^2, \quad (19a)$$

$$D(\theta,\eta,s)=s\omega\left(1+\frac{sm\omega\theta}{2\hbar}\right)\left(1+\frac{s\eta}{2\hbar m\omega}\right), \quad (19b)$$

$$\tilde{E}=\frac{E^2-m^2c^4\pm 2\hbar\omega c^2(1+\frac{sm\omega\theta}{2\hbar})(1+\frac{s\eta}{2\hbar m\omega})}{2c^2}. \quad (19c)$$

It is suitable to introduce polar coordinates, with which we have

$$\left\{-\frac{\hbar^2}{2M}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]+\frac{1}{2}M\Omega^2\rho^2-D\left(-i\hbar\frac{\partial}{\partial\phi}\right)\right\}\psi=\tilde{E}\psi. \quad (20)$$

By taking

$$\psi(\rho,\phi)=\chi(\rho)\exp(ik_\phi\phi), \quad k_\phi=0,\pm1,\pm2,\ldots. \quad (21)$$

And inserting it into (20), we can get the radial equation as

$$\left[\frac{\hbar^2}{2M}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{k_\phi^2}{\rho^2}\right)-\frac{1}{2}M\Omega^2\rho^2+\tilde{\tilde{E}}\right]\psi=0. \quad (22)$$

Here we use the substitution

$$\tilde{\tilde{E}}=\tilde{E}+Dk_\theta\hbar. \quad (23)$$

Introducing a new variable $\xi=M\Omega\rho^2/2\hbar$ and inserting it into (20) we have

$$\xi\frac{d^2\chi(\xi)}{d\xi^2}+\frac{d\chi(\xi)}{d\xi}+\left(\kappa-\xi-\frac{m_l^2}{4\xi}\right)\chi(\xi)=0, \quad (24)$$

where $\kappa=\tilde{\tilde{E}}/\hbar\Omega$. Considering the following trial wave function

$$\chi(\xi)=\exp(-\xi)\xi^{|k_\phi|/2}f(\xi), \quad (25)$$

and inserting it into (22) yields

$$\xi\frac{d^2f(\xi)}{d\xi^2}+(\gamma-2\xi)\frac{df(\xi)}{d\xi}-2\alpha f(\xi)=0, \quad (26)$$

where

$$\gamma = |k_\phi| + 1, \qquad \alpha = \frac{-(\kappa - \gamma)}{2}. \tag{27}$$

If one takes $z = 2\xi$, and inserting it into (26), we obtain the confluent hypergeometric equation as

$$z\frac{d^2 f(z)}{dz^2} + (\gamma - z)\frac{df(z)}{dz} - \alpha f(z) = 0. \tag{28}$$

Its solutions are well-known confluent hypergeometric function type

$$f(z) = AF(\alpha, \gamma, z). \tag{29}$$

Now we obtain the general solutions of wave function ψ_2 as follows

$$\psi(\rho, \phi) = A\exp\left(-\frac{M\Omega\rho^2}{2\hbar} + ik_\phi\phi\right)\rho^{|k_\phi|}F\left(\alpha, \gamma, \frac{M\Omega\rho^2}{\hbar}\right). \tag{30}$$

Considering the boundary conditions that $\rho \to \infty$ lead $\psi \to 0$, we obtain that

$$\alpha = -n, \quad n = 0, 1, 2, \ldots. \tag{31}$$

Substituting (27) into (31) we have

$$\frac{\tilde{\tilde{E}}/\hbar\Omega - |k_\phi| - 1}{2} = n, \quad n = 0, 1, 2, \ldots. \tag{32}$$

Inserting (19b, 19c) and (23), we obtain the energy eigenvalues

$$E^2_{n,m_l} = m^2c^4 + 2mc^2\hbar\Omega(2n + |k_\phi| + 1) - 2\hbar m\omega c^2\left(1 + \frac{sm\omega\theta}{2\hbar}\right)\left(1 + \frac{s\eta}{2\hbar m\omega}\right)(sk_\phi \pm 1). \tag{33}$$

Thus, for the noncommutative effect we can see the energy spectrum of a Dirac oscillator is not degenerate.

The non-relativistic limit can be obtained when considering $E_{n,m_l} = mc^2 + \varepsilon_{n,m_l}$ with $\varepsilon_{n,m_l} \ll mc^2$. The first-order approximation of Taylor expansion is given as

$$E^{\lim}_{n,m_l} \approx \pm mc^2\left(1 + \frac{\hbar\Omega}{mc^2}(2n + |k_\phi| + 1) - \frac{m\omega\hbar(1 + \frac{sm\omega\theta}{2\hbar})(1 + \frac{s\eta}{2\hbar m\omega})(sk_\phi \pm 1)}{m^2c^2}\right). \tag{34}$$

Furthermore, from (27) and (30), it is straightforward to obtain the corresponding total wave function $\Psi_{n,m_l}(\rho, \phi)$ as follows

$$\begin{aligned}\psi_{n,k_\phi,s}(\rho, \phi) &= \begin{pmatrix}\psi_A(\rho,\phi)\\ \psi_B(\rho,\phi)\end{pmatrix}\\ &= B\left(\begin{pmatrix}1\\ G_1\end{pmatrix}F\left(-n, |k_\phi| + 1, \frac{M\Omega\rho^2}{\hbar}\right)\right.\\ &\quad\left.+\begin{pmatrix}0\\ G_2\end{pmatrix}F\left(-n + 1, |k_\phi| + 2, \frac{M\Omega\rho^2}{\hbar}\right)\right),\end{aligned} \tag{35}$$

where

$$G_1 = \frac{c}{E+mc^2}\left[\left(1+\frac{sm\omega\theta}{2\hbar}\right)\left(\frac{sk_\phi+|k_\phi|}{\rho}-\frac{M\Omega\rho}{\hbar}\right) - \left(1+\frac{s\eta}{2m\omega\hbar}\right)im\omega\rho\right](\cos\phi+is\sin\phi)$$
$$= G_1'(\cos\phi+is\sin\phi), \tag{36}$$

$$G_2 = \frac{c}{E+mc^2}\left(1+\frac{sm\omega\theta}{2\hbar}\right)\left(\frac{-n}{|k_\phi|+1}\right)(\cos\phi+is\sin\phi)$$
$$= G_2'(\cos\phi+is\sin\phi). \tag{37}$$

Using the normalization condition

$$\int \bar{\Psi}(\rho,\phi)\beta\Psi(\rho,\phi)\rho d\rho d\phi = 1,$$

we obtain the normalization constant as

$$N = \sqrt{\frac{(M\Omega)^{|k_\phi|+1}}{2\pi\hbar^{|k_\phi|+1}\sum_{q=0}^{n}\{\sum_{p=0}^{n}[\frac{(-n)_q(-n)_p}{q!p!(|k_\phi|+1)_q(|k_\phi|+1)_p}\Gamma(|k_\phi|+q+p+1)(1+I_1+I_2+I_3+I_4)]\}}}, \tag{38}$$

where

$$I_1 = G_1'^{*}G_1'(|k_\phi|+q+p+3)(|k_\phi|+q+p+2), \tag{39a}$$

$$I_2 = (-)G_1'^{*}G_2'(|k_\phi|+q+p+2)\frac{(q-n+1)(|k_\phi|+2)}{(-n+1)(|k_\phi|+q+2)}, \tag{39b}$$

$$I_3 = (-)G_1'G_2'(|k_\phi|+q+p+2)\frac{(p-n+1)(|k_\phi|+2)}{(-n+1)(|k_\phi|+p+2)}, \tag{39c}$$

$$I_4 = G_2'G_2'\frac{(q-n+1)(|k_\phi|+2)(p-n+1)(|k_\phi|+2)}{(-n+1)(|k_\phi|+q+2)(-n+1)(|k_\phi|+p+2)}. \tag{39d}$$

In this limit, the energy eigenvalues are similar as those of the system under action of the magnetic field in commutative space (normal Zeeman effect).

3 Conclusions

We have presented the exact energy spectrum and wave functions of the Dirac oscillator in noncommutative phase space. Therefore, we can conclude that the energy gap was changed by noncommutative effect. Utilizing these different properties from ordinary commutative Dirac oscillator, we hope it is possible to test the noncommutative effect in the future experiments at the level of Quantum mechanics. In addition, we also obtain the non-relativistic limit of the energy spectrum, which are similar as the energy spectrum of the system under action of the magnetic field in commutative space (normal Zeeman effect).

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